

# Gröbner-Shirshov bases for braid groups in Adyan-Thurston generators\*

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**Abstract:** In this paper, we give a Gröbner-Shirshov basis of the braid group  $B_{n+1}$  in Adyan-Thurston generators. We also deal with the braid group of type  $\mathbf{B}_n$ . As results, we obtain a new algorithm for getting the Adyan-Thurston normal form, and a new proof that the braid semigroup  $B_{n+1}^+$  is the subsemigroup in  $B_{n+1}$ .

**Key words:** braid group; Adyan-Thurston generators; Gröbner-Shirshov basis; normal form.

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## 1 Introduction

Artin [2] invented a group  $B_{n+1}$ , the braid group on  $n + 1$  strands

$$B_{n+1} = gp\langle \sigma_1, \dots, \sigma_n \mid \sigma_i \sigma_j = \sigma_j \sigma_i \ (i - 1 > j), \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \rangle$$

and solved the word problem for  $B_{n+1}$ . Markov [17] and Artin [3] found normal form for  $B_{n+1}$  in Artin-Bureau generators

$$s_{i,j}, s_{i,j}^{-1} \ (1 \leq i < j \leq n), \ \sigma_i \ (1 \leq i \leq n),$$

where  $s_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ . Markov and Artin gave another algorithm for the solution of the word problem for  $B_{n+1}$ . Garside [16] found a normal form for  $B_{n+1}$  in Artin-Garside generators

$$\Delta, \Delta^{-1}, \sigma_i \ (1 \leq i \leq n),$$

where  $\Delta = \sigma_1 \sigma_2 \sigma_1 \cdots \sigma_{n-1} \cdots \sigma_1 \sigma_n \cdots \sigma_1$  and used the normal form for the positive solution of the conjugacy problem for  $B_{n+1}$ . Birman-Ko-Lee [5] invented a new presentation:

$$B_{n+1} = gp\langle a_{ts} \ (1 \leq s < t \leq n) \mid R \rangle,$$

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where  $a_{ts} = (\sigma_{t-1} \cdots \sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1} \cdots \sigma_{t-1}^{-1})$  ( $1 \leq s < t \leq n+1$ ) and  $R$  consists of the following relations

$$\begin{cases} a_{ts}a_{rq} = a_{rq}a_{ts}, & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr}, & \text{for } 1 \leq r < s < t \leq n+1 \end{cases}$$

and found a normal form for  $B_{n+1}$  in the new presentation. They used the normal form for another algorithms for the solution of the word and the conjugacy problems for  $B_{n+1}$ .

Bokut-Chainikov-Shum [10] found a Gröbner-Shirshov basis for  $B_{n+1}$  in Artin-Burau generators and as a corollary the Markov-Artin normal form is followed. Bokut-Fong-Ke-Shiao [11] found a Gröbner-Shirshov basis for the braid semigroup

$$B_{n+1}^+ = sgp\langle \sigma_1, \dots, \sigma_n \mid \sigma_i\sigma_j = \sigma_j\sigma_i \ (i-1 > j), \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$$

in the Artin-Garside generators. Using this result, Bokut [8] found Gröbner-Shirshov basis for  $B_{n+1}$  in the Artin-Garside generators. As a corollary, the Garside normal form for  $B_{n+1}$  is followed together with a new algorithm to reach the Garside normal form of a braid. Bokut [9] found a Gröbner-Shirshov basis for  $B_{n+1}$  in the Birman-Ko-Lee generators and hence a new algorithm and a new proof for Birman-Ko-Lee normal form in  $B_{n+1}$ .

Braid group  $B_n$  is a generalization of the symmetric group  $S_n$ , which is the same as Artin group (a generalization of the Coxeter group). The Coxeter graphs  $\mathbf{A}_n, \mathbf{B}_n, \mathbf{D}_n$  (the spherical type) are the same as Dynkin diagrams  $\mathbf{A}_n, \mathbf{B}_n, \mathbf{D}_n$  respectively. Hence, there are also finite types  $\mathbf{A}_n, \mathbf{B}_n, \mathbf{D}_n, \mathbf{G}_2, \mathbf{F}_4, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$  of braid groups.

The following preliminaries are related to the Gröbner-Shirshov bases for associative algebras.

Let  $k$  be a field,  $k\langle X \rangle$  the free associative algebra over  $k$  generated by  $X$  and  $X^*$  the free monoid generated by  $X$ , where the empty word is the identity which is denoted by 1. For a word  $w \in X^*$ , we denote the length of  $w$  by  $|w|$ . Let  $X^*$  be a well ordered set. Let  $f = \alpha\bar{f} + \sum \alpha_i u_i \in k\langle X \rangle$ , where  $\alpha, \alpha_i \in k, \bar{f}, u_i \in X^*$  and  $u_i < \bar{f}$ . Then we call  $\bar{f}$  the leading word and  $f$  monic if  $\bar{f}$  has coefficient 1.

A well ordering  $<$  on  $X^*$  is monomial if it is compatible with the multiplication of words, that is, for  $u, v \in X^*$ , we have

$$u < v \Rightarrow w_1 u w_2 < w_1 v w_2 \text{ for all } w_1, w_2 \in X^*.$$

A standard example of monomial ordering on  $X^*$  is the deg-lex ordering to compare two words first by degree and then lexicographically, where  $X$  is a well ordered set.

Let  $f$  and  $g$  be two monic polynomials in  $k\langle X \rangle$  and  $<$  a well ordering on  $X^*$ . Then there are two kinds of compositions:

(i) If  $w$  is a word such that  $w = \bar{f}b = a\bar{g}$  for some  $a, b \in X^*$  with  $|\bar{f}| + |\bar{g}| > |w|$ , then the polynomial  $(f, g)_w = fb - ag$  is called the intersection composition of  $f$  and  $g$  with respect to  $w$ .

(ii) If  $w = \bar{f} = a\bar{g}b$  for some  $a, b \in X^*$ , then the polynomial  $(f, g)_w = f - agb$  is called the inclusion composition of  $f$  and  $g$  with respect to  $w$ .

In  $(f, g)_w$ ,  $w$  is called the ambiguity of the composition.

Let  $S \subset k\langle X \rangle$  such that every  $s \in S$  is monic. Then the composition  $(f, g)_w$  is called trivial modulo  $(S, w)$  if  $(f, g)_w = \sum \alpha_i a_i s_i b_i$ , where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$  and  $a_i s_i b_i < w$ .

Generally, for  $f, g \in k\langle X \rangle$ ,  $f \equiv g \pmod{(S, w)}$  we mean  $f - g = \sum \alpha_i a_i s_i b_i$ , where every  $\alpha_i \in k$ ,  $s_i \in S$ ,  $a_i, b_i \in X^*$  and  $a_i \bar{s}_i b_i < w$ .

$S$  is called a Gröbner-Shirshov basis in  $k\langle X \rangle$  with respect to the well ordering  $<$  if any composition of polynomials in  $S$  is trivial modulo  $S$ .

The following lemma was first proved by Shirshov [18] for free Lie algebras (with deglex ordering) (see also Bokut [6]). Bokut [7] specialized the approach of Shirshov to associative algebras (see also Bergman [4]). For commutative polynomials, this lemma is known as Buchberger's Theorem (see [13, 14]).

**Composition-Diamond Lemma** Let  $k$  be a field,  $A = k\langle X | S \rangle = k\langle X \rangle / Id(S)$  and  $<$  a monomial ordering on  $X^*$ , where  $Id(S)$  is the ideal of  $k\langle X \rangle$  generated by  $S$ . Then the following statements are equivalent:

- (i)  $S$  is a Gröbner-Shirshov basis.
- (ii)  $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$  for some  $s \in S$  and  $a, b \in X^*$ .
- (iii)  $Irr(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}$  is a  $k$ -basis of the algebra  $A = k\langle X | S \rangle$ .

If a subset  $S$  of  $k\langle X \rangle$  is not a Gröbner-Shirshov basis then one can add to  $S$  all nontrivial compositions of polynomials of  $S$  and continue this process repeatedly in order to have a Gröbner-Shirshov basis  $S^c$  that contains  $S$ . Such a process is called the Shirshov algorithm.

Let  $A = sgp\langle X | S \rangle$  be a semigroup presentation. Then  $S$  is also a subset of  $k\langle X \rangle$  and we can find Gröbner-Shirshov basis  $S^c$ , and  $Irr(S^c)$  is a normal form for  $A$ . We also call  $S^c$  a Gröbner-Shirshov basis of  $A$ .

In this paper, we use the Composition-Diamond lemma to get the Gröbner-Shirshov normal form for the braid group  $B_{n+1}$  in Adyan-Thurston generators. It is exactly the left-greedy forms for braid groups. We also use the same method to deal with the braid group of type  $\mathbf{B}_n$ .

## 2 Gröbner-Shirshov basis of the braid group $B_{n+1}$ in Adyan-Thurston generators

In this section, we will give a Gröbner-Shirshov basis of the braid group  $B_{n+1}$  in Adyan-Thurston generators.

Let  $B_{n+1}$  denote the braid group of type  $\mathbf{A}_n$ . Then

$$B_{n+1} = gp\langle \sigma_1, \dots, \sigma_n \mid \sigma_j \sigma_i = \sigma_i \sigma_j \ (j - 1 > i), \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \rangle.$$

The symmetry group is as follow:

$$S_{n+1} = gp\langle s_1, \dots, s_n \mid s_i^2 = 1, s_j s_i = s_i s_j \ (j - 1 > i), s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i \rangle.$$

Bokut and Shiao found the normal form for  $S_{n+1}$  in the following theorem.

**Theorem 2.1** ([12])  $N = \{s_{1i_1}s_{2i_2}\cdots s_{ni_n} \mid i_j \leq j+1\}$  is the Gröbner-Shirshov normal form for  $S_{n+1}$  in generators  $s_i = (i, i+1)$  relative to the deg-lex ordering, where  $s_{ji} = s_js_{j-1}\cdots s_i$  ( $j \geq i$ ),  $s_{jj+1} = 1$ .  $\square$

Let  $\alpha \in S_{n+1}$  and  $\bar{\alpha} = s_{1i_1}s_{2i_2}\cdots s_{ni_n} \in N$  is the normal form of  $\alpha$ . Define the length of  $\alpha$  as  $|\bar{\alpha}| = l(s_{1i_1}s_{2i_2}\cdots s_{ni_n})$  and  $\alpha \perp \beta$  if  $|\overline{\alpha\beta}| = |\bar{\alpha}| + |\bar{\beta}|$ . Moreover, each  $\bar{\alpha} \in N$  has a unique expression  $\bar{\alpha} = s_{i_1i_{l_1}}s_{i_2i_{l_2}}\cdots s_{i_t i_{l_t}}$ , where each  $s_{i_j i_{l_j}} \neq 1$ . Such a  $t$  is called the breath of  $\alpha$ .

We can easily get the following lemmas.

**Lemma 2.2** Let  $\alpha, \beta, \gamma \in S_{n+1}$ . If  $|\overline{\alpha\beta\gamma}| = |\bar{\alpha}| + |\bar{\beta}| + |\bar{\gamma}|$ , then  $\alpha \perp \beta \perp \gamma$ ,  $\alpha \perp \beta\gamma$  and  $\alpha\beta \perp \gamma$ .  $\square$

**Lemma 2.3** Let  $\alpha, \beta, \gamma \in S_{n+1}$ . If  $\alpha\beta \perp \gamma$  and  $\alpha \perp \beta$ , then  $\alpha \perp \beta\gamma$  and  $\beta \perp \gamma$ .  $\square$

Now, we let

$$B'_{n+1} = gp\langle r(\bar{\alpha}), \alpha \in S_{n+1} \setminus \{1\} \mid r(\bar{\alpha})r(\bar{\beta}) = r(\overline{\alpha\beta}), \alpha \perp \beta \rangle,$$

where  $r(\bar{\alpha})$  means a letter with the index  $\bar{\alpha}$ .

Then  $B_{n+1} \cong B'_{n+1}$ . Indeed, define  $\theta : B_{n+1} \rightarrow B'_{n+1}$ ,  $\sigma_i \mapsto r(s_i)$  and  $\theta' : B'_{n+1} \rightarrow B_{n+1}$ ,  $r(\bar{\alpha}) \mapsto \bar{\alpha}|_{s_i \mapsto \sigma_i}$ . Then two mappings are homomorphisms and  $\theta\theta' = \mathbb{I}_{B'_{n+1}}$ ,  $\theta'\theta = \mathbb{I}_{B_{n+1}}$ . Hence,

$$B_{n+1} = gp\langle r(\bar{\alpha}), \alpha \in S_{n+1} \setminus \{1\} \mid r(\bar{\alpha})r(\bar{\beta}) = r(\overline{\alpha\beta}), \alpha \perp \beta \rangle.$$

Let  $X = \{r(\bar{\alpha}), \alpha \in S_{n+1} \setminus \{1\}\}$ . The generator  $X$  of  $B_{n+1}$  is called Adyan-Thurston generator. It is clear that each  $r(\bar{\alpha})$  corresponds to a positive braid which is non-repeating in Epstein at al's book [15].

Then the positive braid semigroup in generator  $X$  is

$$B_{n+1}^+ = sgp\langle X \mid r(\bar{\alpha})r(\bar{\beta}) = r(\overline{\alpha\beta}), \alpha \perp \beta \rangle.$$

Let  $s_1 < s_2 < \cdots < s_n$ . Define  $r(\bar{\alpha}) < r(\bar{\beta})$  if and only if  $|\bar{\alpha}| > |\bar{\beta}|$  or  $|\bar{\alpha}| = |\bar{\beta}|$ ,  $\bar{\alpha} <_{lex} \bar{\beta}$ . It is clear that such an ordering on  $X$  is well ordered. We will use the deg-lex ordering on  $X^*$  in this section.

**Theorem 2.4** A Gröbner-Shirshov basis of  $B_{n+1}^+$  in Adyan-Thurston generator  $X$  relative to the deg-lex ordering on  $X^*$  is:

$$\begin{aligned} r(\bar{\alpha})r(\bar{\beta}) &= r(\overline{\alpha\beta}), \quad \alpha \perp \beta, \\ r(\bar{\alpha})r(\bar{\beta}\bar{\gamma}) &= r(\overline{\alpha\beta})r(\bar{\gamma}), \quad \alpha \perp \beta \perp \gamma. \end{aligned}$$

**Proof:** The composition of  $r(\bar{\alpha})r(\bar{\beta})$  and  $r(\bar{\beta})r(\bar{\gamma})$  would induce the relation  $r(\bar{\alpha})r(\bar{\beta}\bar{\gamma}) = r(\overline{\alpha\beta})r(\bar{\gamma})$  when  $|\overline{\alpha\beta\gamma}| \neq |\bar{\alpha}| + |\bar{\beta}\bar{\gamma}|$ .

All possible ambiguities of compositions are:

$$\begin{aligned}
w_1 &= r(\overline{\alpha})r(\overline{\beta})r(\overline{\gamma}), \alpha \perp \beta \perp \gamma, \\
w_2 &= r(\overline{\alpha})r(\overline{\beta\gamma})r(\overline{\delta}), \alpha \perp \beta \perp \gamma, \beta\gamma \perp \delta, \\
w_3 &= r(\overline{\alpha})r(\overline{\beta})r(\overline{\gamma\delta}), \alpha \perp \beta \perp \gamma \perp \delta, \\
w_4 &= r(\overline{\alpha})r(\overline{\beta\gamma})r(\overline{\delta\mu}), \alpha \perp \beta \perp \gamma, \beta\gamma \perp \delta \perp \mu.
\end{aligned}$$

We only check the composition  $w_4$ . The others are similar.

Let  $f = r(\overline{\alpha})r(\overline{\beta\gamma}) - r(\overline{\alpha\beta})r(\overline{\gamma})$ ,  $g = r(\overline{\beta\gamma})r(\overline{\delta\mu}) - r(\overline{\beta\gamma\delta})r(\overline{\mu})$ . Then, by Lemma 2.3,  $\gamma \perp \delta$ ,  $\beta \perp \gamma\delta$  and

$$\begin{aligned}
(f, g)_{w_4} &= (r(\overline{\alpha})r(\overline{\beta\gamma}) - r(\overline{\alpha\beta})r(\overline{\gamma}))r(\overline{\delta\mu}) - r(\overline{\alpha})(r(\overline{\beta\gamma})r(\overline{\delta\mu}) - r(\overline{\beta\gamma\delta})r(\overline{\mu})) \\
&= r(\overline{\alpha})r(\overline{\beta\gamma\delta})r(\overline{\mu}) - r(\overline{\alpha\beta})r(\overline{\gamma})r(\overline{\delta\mu}) \\
&\equiv r(\overline{\alpha\beta})r(\overline{\gamma\delta})r(\overline{\mu}) - r(\overline{\alpha\beta})r(\overline{\gamma\delta})r(\overline{\mu}) \\
&\equiv 0.
\end{aligned}$$

Hence the result holds.  $\square$

Let  $\Delta = r(s_{11}s_{21} \cdots s_{n1})$ . Then we have

**Lemma 2.5** ([15])  $r(s_i)\Delta = \Delta r(s_{n+1-i})$ .  $\square$

In  $B_{n+1}$ , the following formulas hold.

- 1)  $(\sigma_{i1}\sigma_{i+11} \cdots \sigma_{n1})(\sigma_{n+1-(i-1)n+1-(i-1)}\sigma_{n+1-(i-2)n+1-(i-1)} \cdots \sigma_{nn+1-(i-1)}) = \Delta;$
- 2)  $(\sigma_{ii_1}\sigma_{jj_1})(\sigma_{j_1-11}\sigma_{i_12}\sigma_{i+22}\sigma_{i+32} \cdots \sigma_{j_22}\sigma_{j+11} \cdots \sigma_{n1}) = \sigma_{i1} \cdots \sigma_{n1};$
- 3)  $(\sigma_{ii_1}\sigma_{jj_1}\sigma_{kk_1})(\sigma_{k_1-11}\sigma_{j_12}\sigma_{i_1+13}\sigma_{i+33}\sigma_{i+43} \cdots \sigma_{j+13}\sigma_{j+22} \cdots \sigma_{k_22}\sigma_{k+11} \cdots \sigma_{n1}) = \sigma_{i1} \cdots \sigma_{n1}.$

**Lemma 2.6** ([1]) For any  $\alpha \in S_{n+1}$ , there exists an  $E_\alpha \in S_{n+1}$  such that in  $B_{n+1}$ ,  $r(\overline{\alpha})r(\overline{E_\alpha}) = \Delta$ .

**Proof:** If  $\alpha = s_i$ , we set  $E_\alpha = s_{11}s_{21} \cdots s_{i-11}s_{i2}s_{i+11} \cdots s_{n1}$ .

If  $|\alpha| \geq 2$ , we prove the result by induction on the breath of  $\alpha$ .

By the above formulas and Lemma 2.2, for any  $\alpha \in S_{n+1} \setminus \{1\}$ , there exists  $E_\alpha \in S_{n+1}$ , such that  $\alpha E_\alpha = s_{11}s_{21} \cdots s_{n1}$  and  $|\overline{\alpha E_\alpha}| = |\overline{\alpha}| + |\overline{E_\alpha}| = n(n+1)/2$ . Hence  $r(\overline{\alpha})r(\overline{E_\alpha}) = \Delta$ ,  $\alpha \perp E_\alpha$ .  $\square$

Now, we can represent the braid group as a semigroup:

$$B_{n+1} = \text{sgp}\langle X, \Delta^{-1} \mid \Delta^\varepsilon \Delta^{-\varepsilon} = 1, \varepsilon = \pm 1, r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}), \alpha \perp \beta \rangle.$$

**Theorem 2.7** *A Gröbner-Shirshov basis of  $B_{n+1}$  in Adyan-Thurston generator  $X$  relative to the deg-lex ordering on  $X^*$  is:*

- 1)  $r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}), \quad \alpha \perp \beta,$
- 2)  $r(\overline{\alpha})r(\overline{\beta\gamma}) = r(\overline{\alpha\beta})r(\overline{\gamma}), \quad \alpha \perp \beta \perp \gamma,$
- 3)  $r(\overline{\alpha})\Delta^\varepsilon = \Delta^\varepsilon r(\overline{\alpha'}), \quad \overline{\alpha'} = \overline{\alpha}|_{s_i \mapsto s_{n+1-i}},$
- 4)  $r(\overline{\alpha\beta})r(\overline{\gamma\mu}) = \Delta r(\overline{\alpha'})r(\overline{\mu}), \quad \alpha \perp \beta \perp \gamma \perp \mu, \quad r(\overline{\beta\gamma}) = \Delta,$
- 5)  $\Delta^\varepsilon \Delta^{-\varepsilon} = 1.$

**Proof:** We will prove that all possible compositions are trivial modulo  $S$ . Denote by  $(i \wedge j)_w$  the composition of the type  $i$ ) and type  $j$ ) with respect to the ambiguity  $w$ . The ambiguities  $w$  of all possible compositions are:

$$\begin{array}{llll}
1 \wedge 1 & r(\overline{\alpha})r(\overline{\beta})r(\overline{\gamma}) & 1 \wedge 2 & r(\overline{\alpha})r(\overline{\beta})r(\overline{\gamma\mu}) \\
2 \wedge 1 & r(\overline{\alpha})r(\overline{\beta\gamma})r(\overline{\mu}) & 2 \wedge 2 & r(\overline{\alpha})r(\overline{\beta\gamma})r(\overline{\mu\nu}) \\
3 \wedge 5 & r(\overline{\alpha})\Delta^\varepsilon \Delta^{-\varepsilon} & 4 \wedge 1 & r(\overline{\alpha\beta})r(\overline{\gamma\mu})r(\overline{\nu}) \\
4 \wedge 4 & r(\overline{\alpha\beta})r(\overline{\gamma\mu})r(\overline{\nu\omega}) & 5 \wedge 5 & \Delta^\varepsilon \Delta^{-\varepsilon} \Delta^\varepsilon
\end{array}
\begin{array}{llll}
1 \wedge 3 & r(\overline{\alpha})r(\overline{\beta})\Delta^\varepsilon & 1 \wedge 4 & r(\overline{\alpha})r(\overline{\beta\gamma})r(\overline{\mu\nu}) \\
2 \wedge 3 & r(\overline{\alpha})r(\overline{\beta\gamma})\Delta^\varepsilon & 2 \wedge 4 & r(\overline{\alpha})r(\overline{\beta\gamma})r(\overline{\mu\nu}) \\
4 \wedge 2 & r(\overline{\alpha\beta})r(\overline{\gamma\mu})r(\overline{\nu\omega}) & 4 \wedge 3 & r(\overline{\alpha\beta})r(\overline{\gamma\mu})\Delta^\varepsilon
\end{array}$$

We only check the composition  $(4 \wedge 4)_w$ . The others are similar. Let  $f = r(\overline{\alpha\beta})r(\overline{\gamma\mu}) - \Delta r(\overline{\alpha'})r(\overline{\mu})$ ,  $g = r(\overline{\gamma\mu})r(\overline{\nu\omega}) - \Delta r(\overline{\gamma'})r(\overline{\omega})$ ,  $w = r(\overline{\alpha\beta})r(\overline{\gamma\mu})r(\overline{\nu\omega})$ , where  $\alpha \perp \beta \perp \gamma \perp \mu \perp \nu \perp \omega$ ,  $r(\overline{\beta\gamma}) = r(\overline{\mu\nu}) = \Delta$ . Then

$$\begin{aligned}
(f, g)_w &= (r(\overline{\alpha\beta})r(\overline{\gamma\mu}) - \Delta r(\overline{\alpha'})r(\overline{\mu}))r(\overline{\nu\omega}) - r(\overline{\alpha\beta})(r(\overline{\gamma\mu})r(\overline{\nu\omega}) - \Delta r(\overline{\gamma'})r(\overline{\omega})) \\
&= r(\overline{\alpha\beta})\Delta r(\overline{\gamma'})r(\overline{\omega}) - \Delta r(\overline{\alpha'})r(\overline{\mu})r(\overline{\nu\omega}) \\
&\equiv \Delta r(\overline{\alpha'\beta'})r(\overline{\gamma'})r(\overline{\omega}) - \Delta r(\overline{\alpha'})\Delta r(\overline{\omega}) \\
&\equiv \Delta r(\overline{\alpha'})\Delta r(\overline{\omega}) - \Delta r(\overline{\alpha'})\Delta r(\overline{\omega}) \\
&\equiv 0.
\end{aligned}$$

Hence the result holds.  $\square$

**Corollary 2.8** *Adyan-Thurston normal forms for  $B_{n+1}$  are  $\Delta^k r(\overline{\alpha_1}) \cdots r(\overline{\alpha_s})$ , where  $k \in \mathbb{Z}$ ,  $r(\overline{\alpha_1}) \cdots r(\overline{\alpha_s})$  is minimal in deg-lex ordering.  $\square$*

**Remark:** Actually, the Adyan-Thurston normal forms for the braid group are exactly the left greedy normal forms in Epstein at al's book [15].

### 3 Gröbner-Shirshov basis of the braid group of type $\mathbf{B}_n$

In this section, we will give a Gröbner-Shirshov basis of the braid group of type  $\mathbf{B}_n$  by using the same method in section 2.

Let  $B(B_{n+1})$  denote the braid group of type  $\mathbf{B}_n$ . Then

$$\begin{aligned}
B(B_{n+1}) &= gp\langle \sigma_1, \dots, \sigma_n \mid \sigma_j \sigma_i = \sigma_i \sigma_j \ (j-1 > i), \ \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \\
&\quad \sigma_n \sigma_{n-1} \sigma_n \sigma_{n-1} = \sigma_{n-1} \sigma_n \sigma_{n-1} \sigma_n \rangle.
\end{aligned}$$

For the same as braid group of type  $\mathbf{A}_n$ , we define

$$G = gp\langle s_1, \dots, s_n \mid s_i^2 = 1, s_j s_i = s_i s_j \ (j - 1 > i), \ s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i, \\ s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n \rangle.$$

Then we can view  $G$  as a semigroup with the same generators and relations as group.

Let  $s_1 < s_2 < \dots < s_n$  and define the deg-lex ordering  $<$  on  $S^*$ , where  $S = \{s_1, \dots, s_n\}$ .

**Lemma 3.1** *A Gröbner Shirshov basis of  $G$  in generator  $S$  relative to the deg-lex ordering on  $S^*$  is:*

- 1)  $s_i^2 = 1 \ (1 \leq i \leq n)$ ,
- 2)  $s_j s_i = s_i s_j \ (j - 1 > i)$ ,
- 3)  $s_{ji} s_j = s_{j-1} s_{ji} \ (1 \leq i < j \leq n - 1)$ ,
- 4)  $s_{nj} s_{ni} = s_{n-1} s_{ni} s_{nj+1} \ (1 \leq i \leq j \leq n - 1)$ .

**Proof:** We will prove that all possible compositions are trivial modulo  $S$ . Denote by  $(i \wedge j)_w$  the composition of the type  $i$ ) and type  $j$ ) with respect to the ambiguity  $w$ . The ambiguities  $w$  of all possible compositions are:

$$\begin{array}{ccccc} 1 \wedge 1 \ s_i^3 & 1 \wedge 2 \ s_j^2 s_i & 1 \wedge 3 \ s_j s_{ji} s_j & 1 \wedge 4 \ s_n s_{nj} s_{ni} & 2 \wedge 1 \ s_j s_i^2 \\ 2 \wedge 2 \ s_k s_j s_i & 2 \wedge 3 \ s_k s_{ji} s_j & 3 \wedge 1 \ s_{ji} s_j^2 & 3 \wedge 2 \ s_{kj} s_k s_i & 3 \wedge 3 \ s_{kj} s_{ki} s_k \\ 4 \wedge 1 \ s_{nj} s_{ni} s_i & 4 \wedge 2 \ s_{nk} s_{nj} s_i & 4 \wedge 3 \ s_{nk} s_{nj} s_i & 4 \wedge 4 \ s_{nk} s_{nj} s_{ni} & \end{array}$$

We only check the composition  $(4 \wedge 4)_w$ . The others are similar. Let  $w = s_{nk} s_{nj} s_{ni}$ ,  $f = s_{nk} s_{nj} - s_{n-1} s_{nj} s_{nk+1}$ ,  $g = s_{nj} s_{ni} - s_{n-1} s_{ni} s_{nj+1}$ , where  $1 \leq i \leq j \leq k \leq n - 1$ . Then

$$\begin{aligned} (f, g)_w &= (s_{nk} s_{nj} - s_{n-1} s_{nj} s_{nk+1}) s_{ni} - s_{nk} (s_{nj} s_{ni} - s_{n-1} s_{ni} s_{nj+1}) \\ &= s_{nk} s_{n-1} s_{ni} s_{nj+1} - s_{n-1} s_{nj} s_{nk+1} s_{ni} \\ &\equiv s_{n-2} s_{nk} s_{ni} s_{nj+1} - s_{n-1} s_{nj} s_{n-1} s_{ni} s_{nk+2} \\ &\equiv s_{n-2} s_{n-1} s_{ni} s_{nk+1} s_{nj+1} - s_{n-1} s_{n-2} s_{nj} s_{ni} s_{nk+2} \\ &\equiv s_{n-2} s_{n-1} s_{ni} s_{n-1} s_{nj+1} s_{nk+2} - s_{n-1} s_{n-2} s_{n-1} s_{ni} s_{nj+1} s_{nk+2} \\ &\equiv s_{n-2} s_{n-1} s_{n-2} s_{ni} s_{nj+1} s_{nk+2} - s_{n-2} s_{n-1} s_{n-2} s_{ni} s_{nj+1} s_{nk+2} \\ &\equiv 0. \end{aligned}$$

Hence the result holds.  $\square$

By using Lemma 3.1 and the Composition-Diamond lemma, we have the following theorem.

**Theorem 3.2**  $N = \{s_{1i_1} s_{2i_2} \dots s_{n-1i_{n-1}} s_{nj_1} \dots s_{nj_k} \mid i_l \leq l + 1, \ 1 \leq j_1 < j_2 < \dots < j_k \leq n, \ k \geq 0\}$  is the Gröbner-Shirshov normal form for  $G$  in generator  $S$  relative to the deg-lex ordering on  $S^*$ , where  $s_{ji} = s_j s_{j-1} \dots s_i \ (j \geq i)$ ,  $s_{jj+1} = 1$ .  $\square$

Similar to the case of the braid group  $B_{n+1}$  in the section 2, we introduce the following notations.

Let  $\alpha \in G$  and

$$\overline{\alpha} = s_{1i_1} s_{2i_2} \cdots s_{n-1i_{n-1}} s_{nj_1} \cdots s_{nj_k} \in N$$

is the normal form of  $\alpha$ . Define the length of  $\alpha$  as  $|\overline{\alpha}| = l(s_{1i_1} s_{2i_2} \cdots s_{n-1i_{n-1}} s_{nj_1} \cdots s_{nj_k})$  and  $\alpha \perp \beta$  if  $|\overline{\alpha\beta}| = |\overline{\alpha}| + |\overline{\beta}|$ . Now, we let

$$B(B'_{n+1}) = gp\langle r(\overline{\alpha}), \alpha \in G \setminus \{1\} \mid r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}), \alpha \perp \beta \rangle.$$

Then  $B(B_{n+1}) \cong B(B'_{n+1})$ . Indeed, define  $\theta : B(B_{n+1}) \rightarrow B(B'_{n+1})$ ,  $\sigma_i \mapsto r(s_i)$  and  $\theta' : B(B'_{n+1}) \rightarrow B(B_{n+1})$ ,  $r(\overline{\alpha}) \mapsto \overline{\alpha}|_{s_i \mapsto \sigma_i}$ . Then two mappings are homomorphisms and  $\theta\theta' = \mathbb{1}_{B(B'_{n+1})}$ ,  $\theta'\theta = \mathbb{1}_{B(B_{n+1})}$ . Hence,

$$B(B_{n+1}) = gp\langle r(\overline{\alpha}), \alpha \in G \setminus \{1\} \mid r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}), \alpha \perp \beta \rangle.$$

Let  $X_1 = \{r(\overline{\alpha}), \alpha \in G \setminus \{1\}\}$ . Then the positive braid semigroup of type  $\mathbf{B}_n$  in generator  $X_1$  is:

$$B(B_{n+1}^+) = sgp\langle X_1 \mid r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}), \alpha \perp \beta \rangle.$$

Define  $r(\overline{\alpha}) < r(\overline{\beta})$  if and only if  $|\overline{\alpha}| > |\overline{\beta}|$  or  $|\overline{\alpha}| = |\overline{\beta}|$ ,  $\overline{\alpha} <_{lex} \overline{\beta}$ .

Similar to Theorem 2.4, we have

**Theorem 3.3** *A Gröbner-Shirshov basis of  $B(B_{n+1}^+)$  in generator  $X_1$  relative to the deg-lex ordering on  $X_1^*$  is:*

$$\begin{aligned} r(\overline{\alpha})r(\overline{\beta}) &= r(\overline{\alpha\beta}), \quad \alpha \perp \beta, \\ r(\overline{\alpha})r(\overline{\beta\gamma}) &= r(\overline{\alpha\beta})r(\overline{\gamma}), \quad \alpha \perp \beta \perp \gamma. \end{aligned} \quad \square$$

Let  $\Delta = r(s_{11}s_{21} \cdots s_{n-11}s_{n1}s_{n2} \cdots s_{nn})$ . Then we have

**Lemma 3.4**  $r(s_i)\Delta = \Delta r(s_i)$ .

**Proof:** We need only to show that in  $B(B_{n+1})$

$$\sigma_i(\sigma_{11}\sigma_{21} \cdots \sigma_{n-11}\sigma_{n1}\sigma_{n2} \cdots \sigma_{nn}) = (\sigma_{11}\sigma_{21} \cdots \sigma_{n-11}\sigma_{n1}\sigma_{n2} \cdots \sigma_{nn})\sigma_i.$$

Suppose  $i = n$ . Then

$$\begin{aligned} & \sigma_n(\sigma_{11}\sigma_{21} \cdots \sigma_{n-11}\sigma_{n1}\sigma_{n2} \cdots \sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21} \cdots \sigma_{n-21})\sigma_{n1}\sigma_{n1} \cdots (\sigma_{n2} \cdots \sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21} \cdots \sigma_{n-21})\sigma_{n-1}\sigma_{n1}\sigma_{n2}(\sigma_{n2}\sigma_{n3} \cdots \sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21} \cdots \sigma_{n-21})\sigma_{n-1}\sigma_{n-2}\sigma_{n1}\sigma_{n2}\sigma_{n3}(\sigma_{n3} \cdots \sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21} \cdots \sigma_{n-21})\sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\sigma_{n1}\sigma_{n2}\sigma_{n3}\sigma_{n4}(\sigma_{n4} \cdots \sigma_{nn}) \\ &= \cdots \\ &= (\sigma_{11}\sigma_{21} \cdots \sigma_{n-21})\sigma_{n-1i}\sigma_{n1} \cdots \sigma_{nn-(i-1)}(\sigma_{nn-(i-1)} \cdots \sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21} \cdots \sigma_{n-21})\sigma_{n-12}\sigma_{n1} \cdots \sigma_{nn-1}(\sigma_{nn-1}\sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21} \cdots \sigma_{n-21})\sigma_{n-12}\sigma_{n1} \cdots \sigma_{nn-2}\sigma_{n-1}\sigma_{nn-1}(\sigma_{nn}\sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21} \cdots \sigma_{n-21}\sigma_{n-11}\sigma_{n1} \cdots \sigma_{nn-1}\sigma_{nn})\sigma_n. \end{aligned}$$



Suppose  $1 \leq i \leq n-1$ . Then

$$\sigma_i(\sigma_{11}\sigma_{21}\cdots\sigma_{n-11}\sigma_{n1}\sigma_{n2}\cdots\sigma_{nn}) = (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11})\sigma_{n-i}(\sigma_{n1}\sigma_{n2}\cdots\sigma_{nn}).$$

Since

$$\begin{aligned} & (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11}\sigma_{n1}\sigma_{n2}\cdots\sigma_{nn})\sigma_i \\ &= (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11}\sigma_{n1}\sigma_{n2}\cdots\sigma_{ni+1})\sigma_i(\sigma_{ni+2}\sigma_{ni+3}\cdots\sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11}\sigma_{n1}\sigma_{n2}\cdots\sigma_{ni}\sigma_{ni})\sigma_i(\sigma_{ni+2}\sigma_{ni+3}\cdots\sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11}\sigma_{n1}\sigma_{n2}\cdots\sigma_{ni-1})\sigma_{n-1}(\sigma_{ni}\sigma_{ni+1}\cdots\sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11}\sigma_{n1}\sigma_{n2}\cdots\sigma_{ni-2})\sigma_{n-2}(\sigma_{ni-1}\sigma_{ni}\cdots\sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11}\sigma_{n1}\sigma_{n2}\cdots\sigma_{ni-3})\sigma_{n-3}(\sigma_{ni-2}\sigma_{ni-1}\cdots\sigma_{nn}) \\ &= \cdots \\ &= (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11}\sigma_{ni-(i-1)})\sigma_{n-(i-1)}(\sigma_{n2}\sigma_{n3}\cdots\sigma_{nn}) \\ &= (\sigma_{11}\sigma_{21}\cdots\sigma_{n-11})\sigma_{n-i}(\sigma_{n1}\sigma_{n2}\sigma_{n3}\cdots\sigma_{nn}), \end{aligned}$$

the result holds.  $\square$

**Lemma 3.5**  $r(s_i)r(E_i) = \Delta$ , where

$$E_i = s_{11}s_{21}\cdots s_{i-11}s_{i2}s_{i+11}\cdots s_{n-11}s_{n1}s_{n2}\cdots s_{nn}, \quad 1 \leq i \leq n-1,$$

$$E_n = s_{11}s_{21}\cdots s_{n1}s_{n2}\cdots s_{nn-1}.$$

**Proof:** By Lemma 3.4,  $s_n(s_{11}s_{21}\cdots s_{n-11}s_{n1}s_{n2}\cdots s_{nn}) = (s_{11}s_{21}\cdots s_{n-11}s_{n1}s_{n2}\cdots s_{nn})s_n$  in  $G$ . Hence,  $s_n E_n = s_{11}s_{21}\cdots s_{n-11}s_{n1}s_{n2}\cdots s_{nn}$ . For  $1 \leq i \leq n-1$ , since  $s_i(s_{11}s_{21}\cdots s_{i-11}s_{i2}s_{i+11}\cdots s_{n-11}) = s_{11}s_{21}\cdots s_{n-11}$ ,  $s_i E_i = s_{11}s_{21}\cdots s_{n-11}s_{n1}s_{n2}\cdots s_{nn}$ . But  $|s_i| + |E_i| = |s_i E_i|$ , we can get  $r(s_i)r(E_i) = \Delta$ .  $\square$

Now, we can represent the braid group as a semigroup:

$$B(B_{n+1}) = \text{sgp}\langle X_1, \Delta^{-1} \mid \Delta^\varepsilon \Delta^{-\varepsilon} = 1, r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}), \alpha \perp \beta \rangle.$$

Similar to the case of the braid group  $B_{n+1}$  in the section 2, we have the following theorem:

**Theorem 3.6** *A Gröbner-Shirshov basis of  $B(B_{n+1})$  in generator  $X_1$  relative to the deg-lex ordering on  $X_1^*$  is:*

$$\begin{aligned} & r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}), \quad \alpha \perp \beta, \\ & r(\overline{\alpha})r(\overline{\beta\gamma}) = r(\overline{\alpha\beta})r(\overline{\gamma}), \quad \alpha \perp \beta \perp \gamma, \\ & r(\overline{\alpha})\Delta^\varepsilon = \Delta^\varepsilon r(\overline{\alpha}), \\ & r(\overline{\alpha\beta})r(\overline{\gamma\mu}) = \Delta r(\overline{\alpha})r(\overline{\mu}), \quad \alpha \perp \beta \perp \gamma \perp \mu, \quad r(\overline{\beta\gamma}) = \Delta, \quad \overline{\alpha} = 1 \text{ or } \overline{\mu} = 1, \\ & \Delta^\varepsilon \Delta^{-\varepsilon} = 1. \end{aligned}$$

**Corollary 3.7** *The normal forms for  $B(B_{n+1})$  are  $\Delta^k r(\overline{\alpha_1}) \cdots r(\overline{\alpha_s}) (k \in \mathbb{Z})$ , where  $r(\overline{\alpha_1}) \cdots r(\overline{\alpha_s})$  is minimal in deg-lex ordering.*

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